Connes duality in Lorentzian geometry

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Abstract

The Connes formula giving the dual description for the distance between points of a Riemannian manifold is extended to the Lorentzian case. It resulted that its validity essentially depends on the global structure of spacetime. The duality principle classifying spacetimes is introduced. The algebraic account of the theory is suggested as a framework for quantization along the lines proposed by Connes. The physical interpretation of the obtained results is discussed.

Introduction

The mathematical account of general relativity is based on the Lorentzian geometry being a model of spacetime. As any model, it needs identification for a physicist in terms of measurable values. In this paper we focus on evaluations of intervals between events, and the measurable entities will be the values of scalar fields.

This work was anticipated by the Connes distance formula for Riemannian manifolds

$$\mathrm{Dist}\,(x, y) = \sup\{f(y) - f(x)\}$$

where the supremum is taken over the smooth functions whose gradient does not exceed 1.

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This formula gives rise to a new paradigm in the account of differential geometry being more sound from the physicist's point of view since it expresses the distance through the values of scalar fields on the manifold. Our goal was to investigate to what extent this formula is applicable in Lorentzian manifolds.

The first observation was that even in the Minkowskian space this formula is no longer valid in its literal form. The reason is that the Cauchy inequality on which the Connes formula is based, does not hold in the Minkowskian space. In section 2 following the Connes' guidelines we managed to obtain an *evaluation* rather than the *expression* for the distance. In 'good' cases, in particular, in the Minkowski spacetime, this evaluation is exact and gives an analog of the Connes formula.

The attempt to generalize it to arbitrary Lorentzian manifolds resulted in building of counterexamples which show the drastic difference between the Riemannian and Lorentzian manifolds. In section 3 the duality principle was introduced in order to point out the class of Lorentzian manifolds being as 'good' as Riemannian ones. It turned out that one can find a 'bad' spacetime even among those conformally equivalent to the Minkowskian one. An example is provided in section 3.

The Connes duality principle play an important rôle in the framework of the so-called 'spectral paradigm' in the account of non-commutative differential geometry [2]. However, the correspondence principle for this theory is corroborated on Riemannian manifolds. Following these lines, in section 4 we show a way to introduce non-comutative Lorentzian geometry.

1 Connes formula

Both the Riemannian distance and Lorentzian interval are based on calculation of the same integral:

$$\int_\gamma \sqrt{ds^2}$$

which is always referred to a pair of points. The intervals (resp., distances) as functions of two points are obtained as extremal values of this integral over all appropriate curves connecting the two points.

There is a remarkable duality to evaluate this integral suggested by Connes for the Riemannian case. Consider it in more detail.

For any two points x, y of a Riemannian manifold $\mathcal{M}$ connected by a smooth curve γ the following evaluation of its length $\ell(\gamma)$ takes place:

$$f(y) - f(x) \leq \sup_{\mathcal{M}} |\nabla f| \cdot \ell(\gamma) \tag{1}$$

based on the Cauchy inequality:

$$(\nabla f, \dot{\gamma}) \leq |\nabla f| \cdot |\dot{\gamma}|$$

So, the distance $\rho(x, y)$ between the points of the manifold satisfies the following inequality

$$\rho(x, y) \geq \sup(f(y) - f(x)),$$

where f ranges over all functions whose gradient does not exceed 1: $|\nabla f| \leq 1$.

It was shown by Connes [3], that as a matter of fact no curves are needed to determine the distance, which may be obtained directly as:

$$\rho(x, y) = \sup_{|\nabla f| \leq 1} (f(y) - f(x)) \tag{2}$$

The physical meaning of this result is the following: we can evaluate the distance between the points measuring the difference of potentials of a scalar field whose intensity is not too high. So, the following *duality principle* takes place in Riemannian geometry:

$$\sup(f(y) - f(x)) = \inf \ell(\gamma)$$

Note that this formula is valid even for non-connected spaces: in this case both sides of the above equality are equal $+\infty$, if we assume, as usually, the infinite value of the infimum when the ranging set is void.

The question arises: can we write down a similar evaluation for the Lorentzian case?

2 Duality inequality in Lorentzian geometry

The Cauchy inequality from which the Riemannian duality principle was derived is no longer valid in the Lorentzian case. Instead, we have the following:

$$(\nabla f, \dot{\gamma})^2 \geq (\nabla f)^2 \cdot (\dot{\gamma})^2 \tag{3}$$

where both ∇f, $\dot{\gamma}$ are non-spacelike:

$$(\nabla f)^2 \geq 0, \qquad (\dot{\gamma})^2 \geq 0$$

If under these circumstances we also have $(\nabla f, \dot{\gamma}) \geq 0$, the inequality (3) reduces to

$$(\nabla f, \dot{\gamma}) \geq |\nabla f| \cdot |\dot{\gamma}|$$

Now let x, y be two points of a Lorentzian manifold $\mathcal{M}$ such there is a causal curve γ going from x to y. Then for any global time function f on $\mathcal{M}$ we immediately obtain the analog of the inequality (1)

$$f(y) - f(x) \geq \inf_{\mathcal{M}} |\nabla f| \cdot \ell(\gamma)$$

Introduce the class $\mathcal{F}(\mathcal{M})$ of global time functions satisfying the following condition:

$$(\nabla f)^2 \geq 1 \tag{4}$$

Then the Lorentzian interval between x and y

$$l(x, y) = \sup \int_{\gamma} \sqrt{ds^2} \tag{5}$$

can be evaluated as follows:

$$f(y) - f(x) \geq l(\gamma)$$

provided the class $\mathcal{F}$ (4) is not empty. Introducing the value

$$L(x, y) = \inf_{f \in \mathcal{F}} (f(y) - f(x))$$

we obtain the following *duality inequality*:

$$l(x, y) \leq L(x, y) \tag{6}$$

It is worthy to mention that this inequality is still meaningful when the points x, y can not be connected by a causal curve. In this case the supremum $l(x, y)$ is taken over the empty set of curves and its value is, as usually, taken to be $-\infty$, that is why the inequality (6) trivially holds.

Let us thoroughly describe this construction in the case when $\mathcal{M}$ is the Minkowskian spacetime. The following proposition holds:

Proposition. Let $\mathcal{M}$ be Minkowskian spacetime. Then $L(a, b) = l(a, b)$ for any pair of points $a, b \in \mathcal{M}$.

Proof. Assume with no loss of generality that $a = 0$. If b is in the future cone of $a = 0$, then $l = (b, b)$ is realized on the segment $[0, b]$. The value of L is achieved on the function $f(x) = (b, x)/\sqrt{(b, b)}$. Let b be a future-directed isotropic vector, then $l = 0$. For any ϵ such that $0 < \epsilon < 1$ consider the function

$$f_\epsilon(x) = \frac{((1-\epsilon)b + \epsilon v, x)}{\sqrt{\epsilon(1-\epsilon)(b, v)}}$$

where v a vector defining the time orientation. The direct calculation shows that $(\nabla f)^2 \geq 1$ and $(\nabla f, v) \geq 0$, that is, $f_\epsilon \in \mathcal{F}$. In the meantime $f(b) = \sqrt{\epsilon(v, b)/(1-\epsilon)}$ which can be made arbitrarily close to zero by appropriate choice of ϵ.

Now let the point b be beyond the future cone of the point 0, therefore they can be separated by a spacelike hyperplane $f_k(x) = (k, x) = 0$ and choose the vector k to be future-directed. Then $f_k(b) < 0$. Since $f_{\lambda k}(b) = \lambda f_k \in \mathcal{M}$, the infimum is $-\infty$. In the meantime $l(0, b) = -\infty$ as well since there are no future-directed non-spacelike curves connecting 0 with b.

$\square$

Remark. Note that if we borrow the definition of $l(a, b)$ from [1], namely $l(a, b) = 0$ for $b \notin J^+(a)$, then the duality principle will not hold even for the Minkowskian case: this was the reason for us to introduce the definition (5).

Consider one more example. Let $\mathcal{M} = S^1 \times \mathbf{R}^3$ be a Minkowskian cylinder where S^1 is the time axis. In this case any two points $x, y \in \mathcal{M}$ can be

connected by an arbitrary long timelike curve, therefore $l(x, y) = +\infty$. In the meantime the class $\mathcal{F}(\mathcal{M})$ is empty (since there is no global time functions), and therefore $L(x, y) = +\infty$. So we see that even in this "pathological" case the duality principle is still valid.

Note that the class $\mathcal{F}(\mathcal{M})$ itself characterizes spacetimes. In general, if the class $\mathcal{F}(\mathcal{M})$ is not empty, the spacetime $\mathcal{M}$ is chronological (it follows immediately from that $\mathcal{F}(\mathcal{M})$ consists of global time functions).

Now we may inquire whether all Lorentzian manifolds are as 'good' as Minkowskian? In the next section we show that the answer is no.

3 Duality principle

In the previous section we have proved the duality inequality (6) which is always true in any Lorentzian manifold. However, unlike the case of Riemannian spaces, this inequality may be strict, which is corroborated by the following example. Let $\mathcal{M}$ be a Minkowskian plane from which a closed segment connecting the points $(1, -1)$ and $(-1, 1)$ is cut out (Fig. 1).

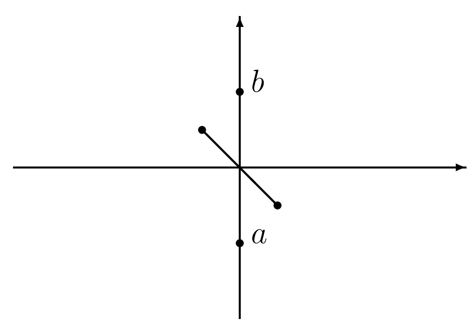

Figure 1: An example where the duality principle is broken.

Consider two points $a = (-2.0)$ and $b = (2, 0)$. They can not be linked by a timelike curve in $\mathcal{M}$, therefore $l(a, b) = -\infty$. Meanwhile the class $\mathcal{F}(\mathcal{M})$ is not empty: it contains at least the restrictions of all such functions defined on the whole Minkowskian plane, thus $L(a, b) < +\infty$. Let us prove that the value $L(a, b)$ is finite, supposing the opposite. If $L(a, b)$ would be equal to

$-\infty$, a function $f \in \mathcal{F}(\mathcal{M})$ should exist for which $f(b) < f(a)$. Consider the behavior of the level line l_b of f passing through the point b. Being spacelike, it can not enter the cone $(-\infty < t < b; |x| \leq |t-b|)$ which contains the point a. From the other hand, the point a must lie in the causal future cone $J_+(l_b)$, which is not the case. So, $L(a,b) \neq l(a,b)$.

Now, specifying the notion of 'good' spacetime, introduce the duality principle.

A Lorentzian manifold $\mathcal{M}$ is said to satisfy the *duality principle* if for any its points x, y

$$L(x,y) = l(x,y)$$

This characterization is global. The example we presented above show that this notion is not hereditary: if we take an open subset of a 'good' manifold it may happen that it will no longer enjoy the duality principle.

Contemplating the above mentioned examples may lead us to an erroneous conclusion that the reason for the duality principle to be broken is when the spacetime manifold is not simply connected. The next example [8] shows that there are manifolds which are simply connected, geodesically convex, admit global chronology but do not enjoy the duality principle.

Let $\mathcal{M}$ be a right semiplane $(-\infty < t < +\infty; x > 0)$ with the metric tensor conformally equivalent to Minkowskian. It is defined in coordinates t, x as follows:

$$g_{ik} = \frac{1}{x} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The example illustrates the problems related with the dual evaluations: it shows that the existence of a global time function does not guarantee the class $\mathcal{F}$ to be non-empty. The spacetime $\mathcal{M}$ evidently admits global time functions such as, for instance, $f(t,x) = t$. However, the following proposition can be proved.

Proposition. The class $\mathcal{F}(\mathcal{M})$ is empty.

Proof. Suppose there is a function $f(x,t)$ satisfying (4) and consider two values A, B ($A < B$) of the function f. The appropriate lines of constant level of f are the graphs of functions $t_A(x), t_B(x)$. Since the derivative $f_t > 0$, we have $t_A(x) < t_B(x)$. These functions are differentiable and their derivatives are bounded: $t'_A, t'_B \leq 1$ (because these lines are always spacelike), therefore they have limits when $x \to 0$. Let us show that these limits are equal.

Consider the difference $B - A$ and evaluate it:

$$B - A = f(t_B(x), x) - f(t_A(x), x) =$$

$$= \int_{t_A(x)}^{t_B(x)} f_t(t,x)dt \geq \frac{1}{\sqrt{x}} \cdot (t_B(x) - t_A(x))$$

where the first factor $1/\sqrt{x}$ is directly obtained from the condition $(\nabla f)^2 \geq 1/x$. So, the limit of $t_B(x) - t_A(x)$ is to be equal 0. Since the values A, B were taken arbitrary, we conclude that all the lines of levels of the global time function f come together to a certain point. Therefore these lines (being spacelike) cannot cover all the manifold $\mathcal{M}$.

□

This proposition shows that the space $\mathcal{M}$ does not support duality principle: we can take two points a, b on a timelike geodesic and calculate the interval $l(a,b)$, while $L(a,b) = +\infty$.

4 Algebraic aspects and quantization

Let us study the dual evaluations from the algebraic point of view. It was pointed out yet by Geroch [6] that the geometrical framework of general relativity can be reformulated in a purely algebraic way. Recall the basic ingredients of Geroch's approach.

The starting object is the algebra $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$, then the vector fields on $\mathcal{M}$ are the derivations of $\mathcal{A}$, that is, the linear mappings $v : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule:

$$v(a \cdot b) = a\, v \cdot b + va \cdot b$$

Denote by $\mathcal{V}$ the set of vector fields on $\mathcal{M}$ (= derivations of $\mathcal{A}$). It is possible to develop tensor calculus along these lines: like in differential geometry,

tensors are appropriate polylinear forms on $\mathcal{V}$. In particular, the metric tensor can be introduced in mere algebraic terms.

The Geroch's viewpoint is in a sense 'pointless' [10]: it contains no points given *ab initio*. However the points are immediately restored as one-dimensional representations of $\mathcal{A}$. For any $x \in \mathcal{M}$ the appropriate representation $\hat{x}$ reads:

$$\hat{x}(a) = a(x) \qquad a \in \mathcal{A}$$

Now let $\mathcal{M}$ be a Riemannian manifold. If we then decide to calculate the distance between two representation x, y in a 'traditional' way we have to introduce such a cumbersome object as continuous curve in the space of representations. It is the result of Connes (2) which lets us stay in the algebraic environment:

$$\rho(x, y) = \sup_{f \in \mathcal{F}} (f(y) - f(x))$$

and the problem now reduces to an algebraic description of the class $\mathcal{F}$ of suitable elements f of the algebra $\mathcal{A}$. The initial Connes' suggestion still refers to points: $\mathcal{F} = \{a \in \mathcal{A} | \, \forall m \in \mathcal{M} \, |\nabla a(m)| \leq 1\}$.

Connes' intention was to build a quantized theory which could incorporate non-commutative algebras as well. For that, the construction of *spectral triple* was suggested [2].

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra of operators $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ and a self-adjoint operator D with compact resolvent in $\mathcal{H}$ such that the commutator $[D, a]$ is bounded for any $a \in \mathcal{A}$ (note that D is not required to be an element of $\mathcal{A}$).

Then for any pair (x, y) of states (= non-negative linear functionals) on $\mathcal{A}$ the distance $d(x, y)$ between x and y may be introduced:

$$d(x, y) = \{|x(a) - y(a)| \, : \, a \in \mathcal{F}\}$$

with the following class of 'test elements' of the algebra $\mathcal{A}$

$$\mathcal{F} = \{a \in \mathcal{A} \, : \, ||[D, a]|| \leq 1\} \tag{7}$$

The suggested construction satisfies the correspondence principle with the Riemannian geometry. Namely, we form the spectral triple with $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$, $\mathcal{H} = \mathcal{L}^2(\mathcal{M}, S)$ — the Hilbert space of square integrable sections of the irreducible spinor bundle over $\mathcal{M}$ and D is the Dirac operator associated with the Levi-Cività connection on $\mathcal{M}$ [9]. Then $d(x, y)$ recovers the Riemannian distance on $\mathcal{M}$ (see, e.g. [2]).

Comparing the definition (7) of the class $\mathcal{F}$ with that used in section 1: $\mathcal{F} = \{a \in \mathcal{A} | \forall m \in \mathcal{M} \ |\nabla a(m)| \le 1\}$ we see that the operator D is a substitute of the gradient. Following [7, 5] the gradient condition (7) can be written in terms of the Laplace operator taking into account that:

$$(\nabla f)^2 = \frac{1}{2}\Delta(f^2) - f\Delta f$$

which restores the metric on $\mathcal{M}$ according to the Connes' duality principle (2) for Riemannian manifolds. However this condition is still checked at every point of $\mathcal{M}$.

We suggest an equivalent algebraic reformulation of (7) with no reference to points. Starting from the notion of the spectrum of an element of algebra [4]:

$$\mathrm{spec}(a) = \{\lambda \in \mathbf{C} | \, a - \lambda \cdot \mathbf{1} \text{ is not invertible}\}$$

and taking into account that the spectrum of the multiplication operator coincides with the domain of the multiplicator we reformulate the Connes' condition $f \in \mathcal{F}$ as

$$\mathrm{spec}(\mathbf{1} - (\nabla f)^2) \quad \text{is non-negative} \tag{8}$$

Within this framework, to pass to Lorentzian case, we simply substitute the Laplacian Δ by the D'Alembertian $\Box$, and the spectral condition (8) is changed to

$$\mathrm{spec}((\nabla f)^2 - \mathbf{1}) \quad \text{is non-negative}$$

which makes it possible to recover the Lorentzian interval provided the duality principle holds.

So we see that the notion of spectral triple is well applicable to develop quantized Lorentzian geometry along the lines of Connes' theory.

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